

Universal Scaling Behavior of Directed Percolation around the Upper Critical Dimension

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In this work we consider the steady state scaling behavior of directed percolation around the upper critical dimension. In particular we determine numerically the order parameter, its fluctuations as well as the susceptibility as a function of the control parameter and the conjugated field. Additionally to the universal scaling functions, several universal amplitude combinations are considered. We compare our results with those of a renormalization group approach.

KEY WORDS: Absorbing phase transition; directed percolation; universal scaling behavior.

1. INTRODUCTION

The concept of universality is one of the most impressive features of continuous phase transitions. It allows to group the great variety of models into a small number of universality classes (see ref. 1 for a recent review). Models within one class share the same critical exponents. Furthermore their corresponding scaling functions become identical close to the critical point. Often, universality classes are also characterized by certain amplitude combinations, which are merely particular values of the scaling functions. The most prominent examples of universal behavior are the coexistence curve of liquid-vapor systems⁽²⁾ and the equation of state in ferromagnetic systems (e.g., refs. 1 and 3). Deciding on a systems universality class by considering the scaling functions instead of critical exponents appears to be less prone to errors in most cases. While for the latter ones

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the variations between different universal classes are often small, the amplitude combinations and therefore the scaling functions may differ significantly (see ref. 4).

Wilson's renormalization group (RG) approach lays the foundation for an understanding of universality.^(5,6) It also yields a tool for computing critical exponents as well as the universal scaling functions. While critical exponents emerge from local properties near a given fixed point, scaling functions require the knowledge of the full RG flow along the trajectories between neighboring fixed points. This illustrates why scaling functions are more *sensitive* than the corresponding exponents.

The RG explains the existence of an upper critical dimension D_c above which the mean-field theory applies, i.e., classical theories, which neglect strong fluctuations and correlations, provide correct estimates to the critical exponents and scaling functions. Below D_c fluctuations become relevant and the mean-field scenario breaks down. At the upper critical dimension the RG equations yield mean-field exponents with logarithmic corrections.^(7,8)

In contrast to equilibrium critical phenomena less is known in the case of non-equilibrium phase transitions. This is due to the fact that a generalized treatment is not possible, lacking an analog to the equilibrium free energy. The rich and often surprising variety of phenomena has to be studied for each system individually. The scaling behavior of directed percolation (DP) is recognized as the paradigm of the critical behavior of several non-equilibrium systems which exhibit a continuous phase transition from an active to an absorbing state (see, e.g., ref. 9). The widespread occurrence of such models describing phenomena in physics, biology, as well as catalytic chemical reactions is reflected by the well known universality hypothesis of Janssen and Grassberger: Short-range interacting models, which exhibit a continuous phase transition into a unique absorbing state belong to the DP universality class, provided they have a one-component order parameter and no additional symmetries.^(10,11) Different universality classes are expected to occur in the presence of additional symmetries, like particle conservation,⁽¹²⁾ particle-hole symmetry (compact directed percolation),⁽¹³⁾ or parity conservation (e.g., branching annihilating random walks with an even number of offsprings).⁽¹⁴⁾ Other model details, such as, e.g., the geometry or shape of a lattice, are expected to have no influence on the scaling behavior in the vicinity of the critical point.

The universality hypothesis still awaits a rigorous proof. Amazingly, numerous simulations suggest that the DP universality class is even larger than expected. It turns out that the hypothesis defines only a sufficient condition but fails to describe the DP class in full generality (see ref. 9 for a detailed discussion). For instance, the pair contact process (PCP) is one of the simplest models with infinitely many absorbing states exhibiting

a continuous phase transition.⁽¹⁵⁾ It was shown that the critical scaling behavior of the one-dimensional PCP is characterized by the same critical exponents^(15,16) as well as by the same universal scaling functions as DP.⁽¹⁷⁾ Thus despite the different structure of the absorbing phase the one-dimensional PCP belongs to the DP universality class. This numerical evidence confirms a corresponding RG-conjecture.⁽¹⁸⁾ But one has to mention that a recently performed RG analysis conjectures a different scaling behavior of both models in higher dimensions.⁽¹⁹⁾

In this work we consider the universal scaling behavior of directed percolation in various dimensions. Whereas most investigations on DP follow the seminal work ref. 20 and thus focus on activity spreading we examine the steady state scaling behavior for $D \geq 2$. We determine the universal scaling functions of the order parameter (i.e., the equation of state) and its fluctuations. Furthermore we consider certain universal amplitude combinations which are related to the order parameter and its susceptibility. These amplitude combinations are immediately related to particular values of the universal scaling functions and are of great experimental interest.⁽⁴⁾ We will see that the numerically obtained universal scaling functions and the related universal amplitude combinations allow a quantitative test of RG-results. The powerful and versatile ϵ -expansion provides estimates of almost all quantities of interest, e.g., the critical exponents and the scaling functions (see, e.g., ref. 21). Unfortunately it is impossible to estimate within this approximation scheme the corresponding error-bars. Thus it is intriguing to compare our results with those of RG analysis.^(22,23)

Furthermore we focus on the phase transition at the upper critical dimension $D_c = 4$. There the usual power-laws are modified by logarithmic corrections. These logarithmic corrections are well established in equilibrium critical phenomena^(7,8) but they have been largely ignored for non-equilibrium phase transitions. Due to the considerable numerical effort, sufficiently accurate simulation data for non-equilibrium systems became available only recently: Investigated systems include self-avoiding random walks,^(24,25) self-organized critical systems,^(26,27) depinning-transitions in disordered media,⁽²⁸⁾ isotropic percolation,⁽²⁹⁾ as well as absorbing phase transitions.⁽³⁰⁾ On the other hand, the numerical advance triggered further analytical RG calculations yielding estimates for the logarithmic correction exponents for the respective systems.^(23,31-33)

The outline of the present paper is as follows: The next section contains the model definition and a description of the method of numerical analysis. In Section 3 we describe the scaling behavior at the critical point and introduce the critical exponents as well as the universal scaling functions. The numerical data of the order parameter and its fluctuations are analyzed in Section 4 below ($D = 2, 3$), above ($D = 5$), and at the upper

critical dimension ($D = 4$). Several amplitude combinations are considered in Section 5. Concluding remarks are given in Section 6.

2. MODEL AND SIMULATIONS

In order to examine the scaling behavior of the D -dimensional DP universality class we consider the directed site percolation process using a generalized Domany–Kinzel automaton.⁽³⁴⁾ It is defined on a $D+1$ -dimensional body centered cubic (bcc) lattice (where time corresponds to the $[0, 0, \dots, 0, 1]$ direction) and evolves by parallel update according to the following rules: A site at time t is occupied with probability p if at least one of its 2^D backward neighboring sites (time $t-1$) is occupied. Otherwise the site remains empty. Furthermore, spontaneous particle creation may take place at all sites with probability p_0 . Directed site percolation corresponds to the choice $p_0 = 0$. The propagation probability p is the control parameter of the phase transition, i.e., below a critical value p_c the activity ceases and the system is trapped forever in the absorbing state (empty lattice). On the other hand a non-zero density of (active) particles ρ_a is found for $p > p_c$. The best estimates of the critical value of directed site percolation on bcc lattices are $p_c = 0.705489(4)$ ⁽³⁵⁾ for $D = 1$ and $p_c = 0.34457(1)$ ⁽³⁶⁾ for $D = 2$.

The order parameter ρ_a of the absorbing phase transition vanishes at the critical point according to

$$\rho_a \propto \delta p^\beta, \quad (1)$$

with $\delta p = (p - p_c)/p_c$. Furthermore the order parameter fluctuations $\Delta\rho_a = L^D(\langle\rho_a^2\rangle - \langle\rho_a\rangle^2)$ diverge as

$$\Delta\rho_a \propto \delta p^{-\gamma'}. \quad (2)$$

The fluctuation exponent γ' obeys the scaling relation $\gamma' = D\nu_\perp - 2\beta$,⁽¹⁶⁾ where ν_\perp describes the divergence of the spatial correlation length at the critical point. The critical behavior of the order parameter is shown in Fig. 1 for $D = 2$. The data are obtained from numerical simulations of systems with periodic boundary conditions. Considering various system sizes L we take care that our results are not affected by finite-size effects. The system is started from a random initial configuration. After a certain transient regime a steady state is reached, which is characterized by the average particle density ρ_a and its fluctuations $\Delta\rho_a$.

Similar to equilibrium phase transitions it is possible in DP to apply an external field h that is conjugated to the order parameter. Being

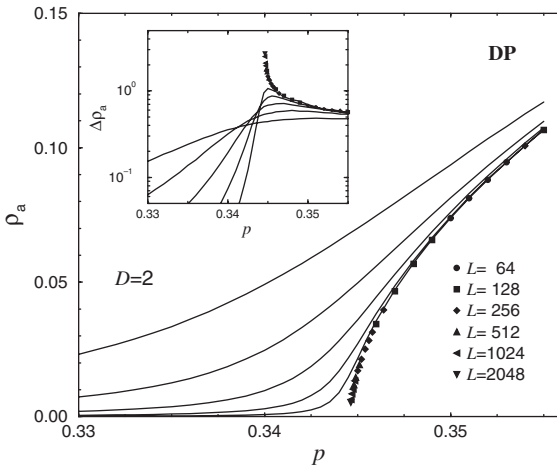


Fig. 1. The two-dimensional directed percolation order parameter ρ_a as a function of the particle density for zero field (symbols) and for various values of the external field ($h = 3 \times 10^{-4}, 10^{-4}, 2 \times 10^{-5}, 5 \times 10^{-6}, 10^{-6}$) (lines). The inset displays the order parameter fluctuations $\Delta\rho_a$ for zero field (symbols) and for various values of the external field h (lines).

a conjugated field it has to destroy the absorbing phase and the corresponding linear response function has to diverge at the critical point i.e.,

$$\chi_a = \frac{\partial \rho_a}{\partial h} \rightarrow \infty. \tag{3}$$

In DP the external field is implemented^(9, 17) as a spontaneous creation of particles (i.e., $p_0 = h > 0$). Clearly, the absorbing state and thus the phase transition are destroyed. Figure 1 shows how the external field results in a smoothing of the zero-field order parameter curve. The inset displays that the fluctuations are peaked for finite fields. Approaching the transition point ($h \rightarrow 0$) this peak becomes a divergence signalling the critical point.

3. UNIVERSAL SCALING FORMS

Sufficiently close to the critical point the order parameter, its fluctuations, as well as the order parameter susceptibility can be described by generalized homogeneous functions

$$\rho_a(\delta p, h) \sim \lambda^{-\beta} \tilde{R}(a_p \delta p \lambda, a_h h \lambda^\sigma), \tag{4}$$

$$a_A \Delta\rho_a(\delta p, h) \sim \lambda^\gamma \tilde{D}(a_p \delta p \lambda, a_h h \lambda^\sigma), \tag{5}$$

$$a_\chi \chi_a(\delta p, h) \sim \lambda^\gamma \tilde{C}(a_p \delta p \lambda, a_h h \lambda^\sigma). \tag{6}$$

Note that these scaling forms are valid for $D \neq D_c$. At the upper critical dimension D_c they have to be modified by logarithmic corrections.⁽³⁰⁾ Taking into consideration that the susceptibility is defined as the derivative of the order parameter with respect to the conjugated field [Eq. (3)] we find $\tilde{C}(x, y) = \partial_y \tilde{R}(x, y)$, $a_\chi = a_h^{-1}$, as well as the Widom scaling law

$$\gamma = \sigma - \beta. \quad (7)$$

The universal scaling functions \tilde{R} , \tilde{D} , and \tilde{C} are identical for all models belonging to a given universality class whereas all non-universal system-dependent details (e.g., the lattice structure, range of interactions, the update scheme, etc.) are contained in the so-called non-universal metric factors a_p , a_h , and a_Δ .⁽³⁷⁾ The universal scaling functions can be normalized by the conditions $\tilde{R}(1, 0) = \tilde{R}(0, 1) = \tilde{D}(0, 1) = 1$. In that case the non-universal metric factors are determined by the amplitudes of the corresponding power-laws

$$\rho_a(\delta p, h = 0) \sim (a_p \delta p)^\beta, \quad (8)$$

$$\rho_a(\delta p = 0, h) \sim (a_h h)^{\beta/\sigma}, \quad (9)$$

$$a_\Delta \Delta \rho_a(\delta p = 0, h) \sim (a_h h)^{-\gamma'/\sigma}. \quad (10)$$

Furthermore we just mention that $\tilde{C}(0, 1) = \beta/\sigma$, following trivially from the definition of the susceptibility.

Usually, an analytical expression for the scaling functions is only known above D_c , where mean-field theories apply. In the case of directed percolation the mean-field scaling functions are given by (see, e.g., ref. 38)

$$\tilde{R}_{\text{MF}}(x, y) = \frac{x}{2} + \sqrt{y + \left(\frac{x}{2}\right)^2}, \quad (11)$$

$$\tilde{D}_{\text{MF}}(x, y) = \frac{\tilde{R}_{\text{MF}}(x, y)}{\sqrt{y + (x/2)^2}}, \quad (12)$$

$$\tilde{C}_{\text{MF}}(x, y) = \frac{1}{2\sqrt{y + (x/2)^2}}, \quad (13)$$

i.e., the mean-field exponents are $\beta_{\text{MF}} = 1$, $\sigma_{\text{MF}} = 2$, $\gamma_{\text{MF}} = 1$, and $\gamma'_{\text{MF}} = 0$ (corresponding to a finite jump of the fluctuations). Below D_c the universal scaling functions depend on dimensionality and are unknown due to a lack of analytical solutions. In this case the scaling functions have to be determined numerically or via approximation schemes, e.g., series expansions or ϵ -expansion of RG approaches.

In case of the mean-field solution ($\gamma'_{\text{MF}} = 0$) the scaling form of the fluctuations [Eq. (5)] reduces to

$$a_\Delta \Delta \rho_a(\delta p, h) \sim \tilde{D}(a_p \delta p \lambda, a_h h \lambda^\sigma). \quad (14)$$

Some interesting properties of the universal scaling function \tilde{D} can be derived from this form. The non-universal metric factor a_Δ is determined by

$$a_\Delta = \frac{1}{\Delta \rho_a(\delta p = 0, h)} \quad (15)$$

using that $\tilde{D}(0, 1) = 1$. The value $\tilde{D}(1, 0)$ is then given by

$$\tilde{D}(1, 0) = \frac{\Delta \rho_a(\delta p, h = 0)}{\Delta \rho_a(\delta p = 0, h)}. \quad (16)$$

Finally, it is worth mentioning that the mean-field scaling function \tilde{D} fulfills the symmetries

$$\tilde{D}(1, x) = \tilde{D}(x^{-1/\sigma}, 1) \quad (17)$$

$$\tilde{D}(x, 1) = \tilde{D}(1, x^{-\sigma}) \quad (18)$$

for all positive x . In particular we obtain for $x \rightarrow 0$ $\tilde{D}(1, 0) = \tilde{D}(\infty, 1)$ and $\tilde{D}(0, 1) = \tilde{D}(1, \infty)$, respectively.

4. EQUATION OF STATE AND FLUCTUATIONS

4.1. Below the Upper Critical Dimension

The scaling forms Eqs. (4)–(6) imply that curves corresponding to different values of the conjugated field collapse to the universal functions $\tilde{R}(x, 1)$, $\tilde{D}(x, 1)$, $\tilde{C}(x, 1)$, if $\rho_a (a_h h)^{-\beta/\sigma}$, $a_\Delta \Delta \rho_a (a_h h)^{\gamma/\sigma}$, and $a_\chi \chi_a (a_h h)^{\gamma/\sigma}$ are considered as functions of the rescaled control parameter $a_p \delta p (a_h h)^{-1/\sigma}$. In a first step, the non-universal metric factors a_p , a_h , a_Δ are obtained from measuring the power-laws Eqs. (8)–(10) (see Table II). Here, the best known estimates for critical exponents, as given in Table I, are used.

Subsequently, the rescaled order parameter and its fluctuations as a function of the rescaled control parameter are plotted for two- and three-dimensional DP (Figs. 2 and 3). A convincing data collapse is achieved, confirming the scaling ansatz as well as the values of the critical exponents.

Table I. The Critical Exponents of Directed Percolation for Various Dimensions D . The One-Dimensional Values Were Obtained in a Famous Series Expansion by Jensen.⁽⁴⁹⁾ For $D=2$ and $D=3$ the Authors Investigated Activity Spreading and the Presented Exponents Are Derived via Scaling Relations. A Complete List of all Critical Exponents of DP can be Found in Ref. 9. The Symbol * Denotes Logarithmic Corrections to the Power-Law Behavior

D	1 ⁽⁴⁹⁾	2 ⁽⁵⁰⁾	3 ⁽⁵¹⁾	4	MF
β	0.276486(8)	0.584(4)	0.81(1)	1*	1
σ	2.554216(13)	2.18(1)	2.04(2)	2*	2
γ'	0.543882(16)	0.300(11)	0.123(25)	0*	0

Besides the universal scaling function $\tilde{R}(x, 1)$ the corresponding curve of an ϵ -expansion obtained from a renormalization group analysis is shown in Figs. 2 and 3. Using the parametric representation^(39,40) of the absorbing phase transition, Janssen *et al.* showed that the equation of state is given by the remarkably simple scaling function⁽²²⁾

$$H(\theta) = \theta(2 - \theta) + \mathcal{O}(\epsilon^3), \quad (19)$$

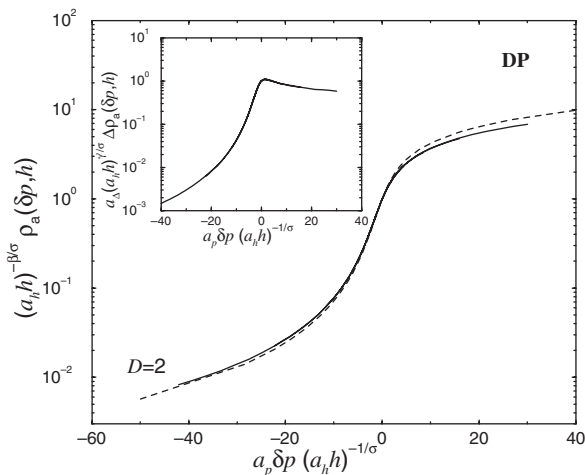


Fig. 2. The universal scaling plots of the order parameter and its fluctuations (inset) for $D = 2$. The dashed line corresponds to an ϵ -expansion of a RG approach.⁽²²⁾

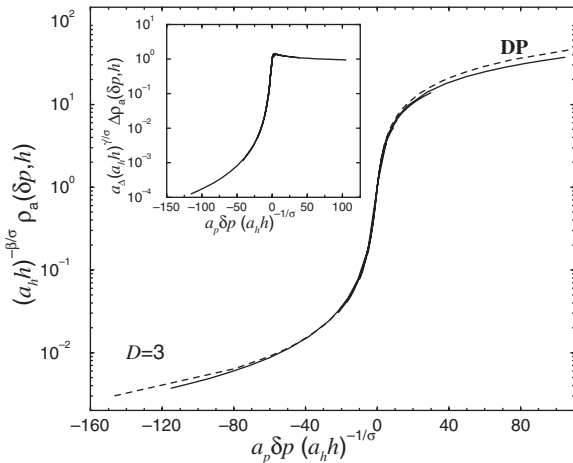


Fig. 3. The universal scaling plots of the order parameter and its fluctuations (inset) for $D=3$ and for various values of the external field ($h = 3 \times 10^{-4}, 10^{-4}, 2 \times 10^{-5}, 4 \times 10^{-6}, 5 \times 10^{-7}$). The dashed line corresponds to an ϵ -expansion of a RG approach.⁽²²⁾

where ϵ denotes the distance to the upper critical dimension $D_c = 4$, i.e., $\epsilon = D_c - D$. Here the scaling behavior of the quantities ρ_a , δp , and h is transformed to the variables R and θ through the relations

$$b \delta p = R(1 - \theta), \quad \rho_a = R^\beta \frac{\theta}{2}. \tag{20}$$

The equation of state is given by

$$ah = \left(\frac{R^\beta}{2} \right)^\delta H(\theta) \tag{21}$$

with the metric factors a and b . The whole phase diagram is described by the parameter range $R \geq 0$ and $\theta \in [0, 2]$. In Figs. 2 and 3 a comparison between the numerically obtained scaling functions and the analytical result of Eqs. (19)–(21) is made. The RG-data differ slightly from the universal function. As expected the differences decrease with increasing dimension and are especially strong in $D = 1$.⁽¹⁷⁾ This point is discussed in detail below.

4.2. Above the Upper Critical Dimension

Above the upper critical dimension the scaling behavior of a phase transition equals the scaling behavior of the corresponding mean-field

solution [Eqs. (11)–(13)]. Plotting $\rho_a/\sqrt{a_h h}$ as a function of $a_p \delta p/\sqrt{a_h h}$, the numerical data should collapse to the universal scaling function

$$\tilde{R}_{\text{MF}}(x, 1) = \frac{x}{2} + \sqrt{1 + \left(\frac{x}{2}\right)^2} \quad (22)$$

with the scaling variable $x = a_p \delta p/\sqrt{a_h h}$. In Fig. 4 we plot the corresponding rescaled data of the five-dimensional model. A perfect collapse of the numerical data and $\tilde{R}(x, 1)$ is obtained. This is a confirmation of the RG-result $D_c = 4$.^(41, 42) To the best of our knowledge no numerical evidence that five-dimensional DP exhibits mean-field scaling behavior was published so far.

The rescaled fluctuation data is presented in Fig. 4. As for the universal order parameter, the data of the fluctuations are in agreement with the corresponding universal mean-field scaling function

$$\tilde{D}_{\text{MF}}(x, 1) = 1 + \frac{x}{2\sqrt{1 + (x/2)^2}}. \quad (23)$$

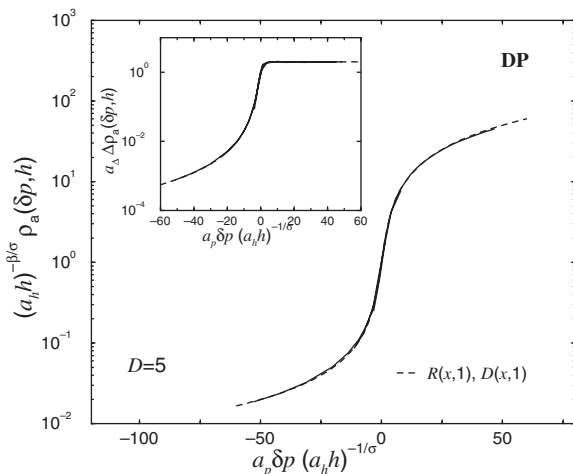


Fig. 4. The universal scaling plots of the order parameter and its fluctuations (inset) for $D = 5$ and for various values of the external field ($h = 5 \times 10^{-5}$, 7×10^{-5} , 10^{-6} , 7×10^{-7}). The dashed lines correspond to the mean-field solutions $\tilde{R}_{\text{MF}}(x, 1)$ and $\tilde{D}_{\text{MF}}(x, 1)$ [see Eqs. (22) and (23)].

4.3. At the Upper Critical Dimension

At the upper critical dimension $D_c = 4$ the scaling behavior is governed by the mean-field exponents modified by logarithmic corrections. For instance the order parameter obeys in leading order

$$\rho_a(\delta p, h = 0) \propto \delta p |\ln \delta p|^B, \quad (24)$$

$$\rho_a(\delta p = 0, h) \propto \sqrt{h} |\ln h|^\Sigma. \quad (25)$$

The logarithmic correction exponents B and Σ are characteristic features of the whole universality class similar to the usual critical exponents. Numerous theoretical, numerical, as well as experimental investigations of critical systems at D_c have been performed (see for instance, refs. 23, 26–28, 32, 33, 43–47)). Logarithmic corrections make the data analysis quite difficult. Hence most investigations are focused on the determination of the correction exponents only, lacking the determination of the scaling functions at D_c .

Recently, a method of analysis was developed to determine the universal scaling functions at the upper critical dimension.⁽³⁰⁾ In this work the authors use the phenomenological scaling ansatz (all terms in leading order)

$$a_a \rho_a(\delta p, h) \sim \lambda^{-\beta_{MF}} |\ln \lambda|^l \tilde{R}(a_p \delta p \lambda |\ln \lambda|^b, a_h h \lambda^{\sigma_{MF}} |\ln \lambda|^s), \quad (26)$$

with $\beta_{MF} = 1$ and $\sigma_{MF} = 2$. Therefore, the order parameter at zero field ($h = 0$) and at the critical point ($\delta p = 0$) are given in leading order by

$$a_a \rho_a(\delta p, h = 0) \sim a_p \delta p |\ln a_p \delta p|^B \tilde{R}(1, 0), \quad (27)$$

$$a_a \rho_a(\delta p = 0, h) \sim \sqrt{a_h h} |\ln \sqrt{a_h h}|^\Sigma \tilde{R}(0, 1) \quad (28)$$

with $B = b + l$ and $\Sigma = s/2 + l$. Similar to the case $D \neq D_c$ the normalization $\tilde{R}(0, 1) = \tilde{R}(1, 0) = 1$ was used. According to the ansatz Eq. (26) the scaling behavior of the equation of state is given in leading order by

$$a_a \rho_a(\delta p, h) \sim \sqrt{a_h h} |\ln \sqrt{a_h h}|^\Sigma \tilde{R}(x, 1) \quad (29)$$

where the scaling argument is given by

$$x = a_p \delta p \sqrt{a_h h}^{-1} |\ln \sqrt{a_h h}|^\Xi \quad (30)$$

with $\Xi = b - s/2 = B - \Sigma$.

In case of directed percolation it is possible to confirm the scaling ansatz Eq. (26) by a RG-approach.⁽²³⁾ In particular the logarithmic correction exponents are given by $l = 7/12$, $b = -1/4$, and $s = -1/2$. Thus the scaling behavior of the equation of state is determined by the logarithmic correction exponents⁽²³⁾

$$B = \Sigma = 1/3, \quad \mathcal{E} = 0. \quad (31)$$

It is worth mentioning that in contrast to the RG results below the upper critical dimension the logarithmic correction exponents do not rely on approximation schemes like ϵ - or $1/n$ -expansions. Within the RG theory they are exact results.

Similarly to the order parameter the following form is used for its fluctuations⁽³⁰⁾

$$a_A \Delta \rho_a(\delta p, h) \sim \lambda^{\gamma'} |\ln \lambda|^k \tilde{D}(a_p \delta p \lambda |\ln \lambda|^b, a_h h \lambda^{-\sigma} |\ln \lambda|^s). \quad (32)$$

Using the mean-field value $\gamma' = 0$ and taking into account that numerical simulations show that the fluctuations remain finite at the critical point (i.e., $k = 0$) the scaling function

$$a_A \Delta \rho_a(\delta p, h) \sim \tilde{D}(x, 1) \quad (33)$$

is obtained, where the scaling argument x is given by Eq. (30) with $\mathcal{E} = 0$. The non-universal metric factor a_A is determined again by the condition $\tilde{D}(0, 1) = 1$.

Thus the scaling behavior of the order parameter and its fluctuations at D_c is determined by two exponents ($B = 1/3$ and $\Sigma = 1/3$) and four unknown non-universal metric factors (a_a, a_p, a_h, a_A). Following⁽³⁰⁾ we determine these values in our analysis by several conditions which are applied simultaneously: first, both the rescaled equation of state and the rescaled order parameter fluctuations have to collapse to the universal functions $\tilde{R}(x, 1)$ and $\tilde{D}(x, 1)$. Second, the order parameter behavior at zero field and at the critical density are asymptotically given by the simple function $f(x) = x$ when plotting $[a_a \rho_a(\delta p, 0)/a_p \delta p]^{1/B}$ vs. $|\ln a_p \delta p|$ and $[a_a \rho_a(0, h)/\sqrt{a_h h}]^{1/\Sigma}$ vs. $|\ln \sqrt{a_h h}|$, respectively. Applying this analysis we observe convincing results for $B = \Sigma = 1/3$, $\mathcal{E} = 0$, and for the values of the non-universal metric factors listed in Table II. The corresponding plots are presented in Fig. 5.

Table II. The Non-Universal Quantities for Site Directed Percolation on a bcc Lattice for Various Dimensions. The Uncertainty of the Metric Factors Is Less than 7%. The Values for $D=1$ Are Obtained From a Previous Work.⁽¹⁷⁾

D	p_c	a_a	a_p	a_h	a_d
1	0.705489 ± 0.000004		2.498	0.114	9.382
2	0.344575 ± 0.000015		0.795	0.186	9.016
3	0.160950 ± 0.000030		0.417	0.328	11.91
4	0.075582 ± 0.000017	14.70	3.055	59.80	19.19
5	0.035967 ± 0.000023		0.114	0.174	42.49

5. UNIVERSAL AMPLITUDE COMBINATIONS

In the following we consider several universal amplitude combinations (see ref. 4 for an excellent review). As pointed out in ref. 4, these amplitude combinations are very useful in order to identify the universality class of a phase transition since the amplitude combinations vary more widely than the corresponding critical exponents. Furthermore, the measurement

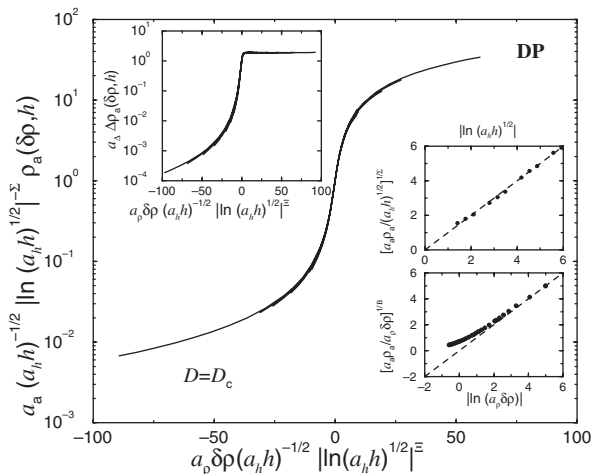


Fig. 5. The universal scaling plots of the order parameter and its fluctuations (upper left inset) at the upper critical dimension $D_c = 4$ for various values of the external field ($h = 5 \times 10^{-5}, 2 \times 10^{-5}, 8 \times 10^{-6}, 4 \times 10^{-6}, 2 \times 10^{-6}$). The logarithmic correction exponents are given by $b = \Sigma = 1/3$ ⁽²³⁾ and $\mathcal{E} = 0$. The right insets show the order parameter at the critical density and for zero field, respectively. The order parameter is rescaled according to Eqs. (27) and (28). Approaching the transition point ($h \rightarrow 0$ and $\delta p \rightarrow 0$) the data tend to the function $f(x) = x$ (dashed lines) as required.

of amplitude combinations in experiments or simulations yields a reliable test for theoretical predictions. In particular, estimates of amplitude combinations are provided by RG approximation schemes like ϵ - or $1/n$ -expansions.

Usually numerical investigations focus on amplitude combinations arising from finite-size scaling analysis. A well known example is the value of Binder's fourth order cumulant at criticality (see, e.g., ref. 48). Instead of those finite-size properties we continue to focus our attention to bulk critical behavior since bulk amplitude combinations are of great experimental interest. Furthermore, they can be compared to RG-results.⁽²²⁾

The susceptibility diverges as

$$\chi(\delta p > 0, h = 0) \sim a_{\chi,+} \delta p^{-\gamma}, \quad (34)$$

$$\chi(\delta p < 0, h = 0) \sim a_{\chi,-} (-\delta p)^{-\gamma}, \quad (35)$$

if the critical point is approached from above and below, respectively. The amplitude ratio

$$\frac{\chi(\delta p > 0, h = 0)}{\chi(\delta p < 0, h = 0)} = \frac{a_{\chi,+}}{a_{\chi,-}} \quad (36)$$

is a universal quantity similar to the critical exponents, i.e., all systems belonging to a given universality class are characterized by the same value $a_{\chi,+}/a_{\chi,-}$. This can be seen from Eq. (6). Setting $a_p |\delta p|^\lambda = 1$ yields

$$\frac{\chi(\delta p > 0, h)}{\chi(\delta p < 0, h)} = \frac{\tilde{C}(+1, x)}{\tilde{C}(-1, x)} \quad (37)$$

with $x = a_h h |a_p \delta p|^{-\sigma}$. Obviously this is a universal quantity for all values of the scaling variable x . In particular it equals the ratio $a_{\chi,+}/a_{\chi,-}$ for $x \rightarrow 0$, i.e., vanishing external field. In general, universal amplitude combinations are related to particular values of the universal scaling functions.

In Fig. 6 the universal susceptibility ratio Eq. (37) is shown for various dimensions. The corresponding data saturates for $x \rightarrow 0$. Our estimates for the amplitude ratios $\tilde{C}(+1, 0)/\tilde{C}(-1, 0)$ are 0.033 ± 0.004 for $D = 1$, 0.25 ± 0.01 for $D = 2$, as well as 0.65 ± 0.03 for $D = 3$. In case of five-dimensional DP the amplitude ratio is constant, as predicted from the mean-field behavior

$$\frac{\tilde{C}_{\text{MF}}(+1, x)}{\tilde{C}_{\text{MF}}(-1, x)} = 1 \quad (38)$$

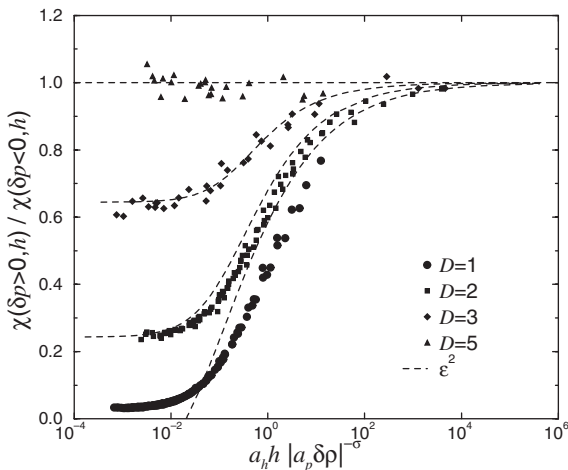


Fig. 6. The universal scaling function $\tilde{C}(1, x)/\tilde{C}(-1, x)$ for various dimensions. The dashed lines correspond to an ϵ -expansion of a RG approach.⁽²²⁾ The universal amplitude $\tilde{C}(1, 0)/\tilde{C}(-1, 0)$ is obtained from the extrapolation $a_h h |a_p \delta p|^{-\sigma} \rightarrow 0$.

for all x . The behavior of the ratio $\tilde{C}(+1, x)/\tilde{C}(-1, x)$ for $D < D_c$ reflects the crossover from mean-field to non mean-field behavior. Far away from the transition point, the critical fluctuations are suppressed and the behavior of the system is well described by the mean-field solution [Eq. (38)]. Approaching criticality the critical fluctuations increase and a crossover to the D -dimensional behavior takes place.

In the already mentioned work,⁽²²⁾ Janssen *et al.* calculated the steady state scaling behavior of DP within a RG approach. In particular they obtained for the susceptibility amplitude ratio

$$\frac{\tilde{C}(+1, 0)}{\tilde{C}(-1, 0)} = 1 - \frac{\epsilon}{3} \left[1 - \left(\frac{11}{288} - \frac{53}{144} \ln \frac{4}{3} \right) \epsilon + \mathcal{O}(\epsilon^2) \right] \quad (39)$$

leading to $-0.2030\dots$ for $D = 1$, $0.2430\dots$ for $D = 2$, $0.6441\dots$ for $D = 3$, Except for the unphysical one-dimensional result these values agree well with our numerical estimates.

Furthermore the parametric representation of the susceptibility was derived in ref. 22 and it is straight forward to calculate the universal ratio Eq. (37). The results are plotted for various dimensions in Fig. 6. It is instructive to compare these results with the numerical data since the theoretical curve reflects the accuracy of the RG estimations of all three quantities, the exponent, the scaling function, as well as the non-universal

metric factors. All quantities are well approximated for the three-dimensional model. In the two-dimensional case we observe a horizontal shift between the numerical data and the RG-estimates. Thus the RG-approach yields good estimates for the exponents and the scaling function but the metric factors are of significantly less quality. For $D=1$ the ϵ^2 -approximation does not provide appropriate estimates of the DP scaling behavior. Thus higher orders than $\mathcal{O}(\epsilon^2)$ are necessary to describe the scaling behavior of directed percolation in low dimensions.

Analogous to the susceptibility the universal amplitude ratio of the fluctuations is given by

$$\frac{\Delta\rho_a(\delta p > 0, h)}{\Delta\rho_a(\delta p < 0, h)} = \frac{\tilde{D}(+1, x)}{\tilde{D}(-1, x)} \quad (40)$$

with $x = a_h h |a_p \delta p|^{-\sigma}$. In the case of absorbing phase transitions this ratio diverges for vanishing field. For $\delta p < 0$ the order parameter fluctuations are zero (absorbing state) for vanishing field whereas the fluctuations remain finite above the transition ($\delta p > 0$). Thus absorbing phase transitions are generally characterized by

$$\frac{\tilde{D}(+1, 0)}{\tilde{D}(-1, 0)} \rightarrow \infty. \quad (41)$$

In Fig. 7 we plot the fluctuation ratio [Eq. (40)] as a function of the scaling variable $a_h h |a_p \delta p|^{-\sigma}$ for various dimensions. We observe in all cases that the fluctuation ratios diverge for $x \rightarrow 0$. Only the one-dimensional system exhibits a particular behavior characterized by the minimum of the corresponding curve. The origin of this behavior is that for $D=2, 3$ the universal scaling function $\tilde{D}(x, 1)$ exhibits a maximum for $x > 0$, whereas for $D=1$ it is located at $x < 0$ (see Fig. 8 in ref. 17).

In the five-dimensional model we observe a perfect agreement with the mean-field behavior

$$\frac{\tilde{D}_{\text{MF}}(+1, x)}{\tilde{D}_{\text{MF}}(-1, x)} = \frac{1 + \sqrt{1 + 4x}}{-1 + \sqrt{1 + 4x}} \xrightarrow{x \rightarrow 0} \frac{1 + 2x}{2x}. \quad (42)$$

Surprisingly, the two- and three-dimensional data are also well approximated by this formula provided that one performs a simple rescaling ($x \mapsto a_D x$) which results in Fig. 7 in a horizontal shift of the data. We suppose that this behavior could be explained by a RG-analysis of the fluctuations.

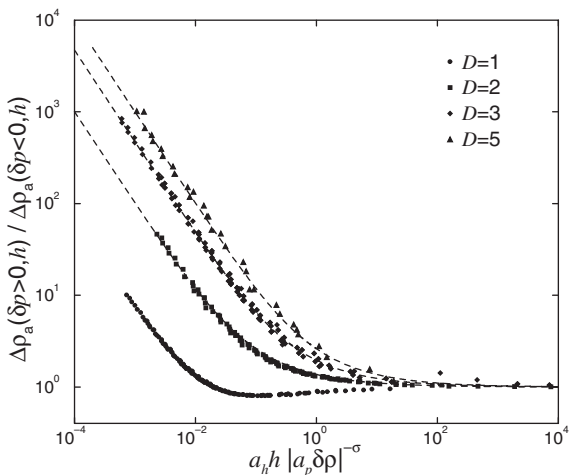


Fig. 7. The universal scaling function $\tilde{D}(1, x)/\tilde{D}(-1, x)$ for various dimensions. The dashed line corresponds to the mean-field scaling behavior. For $D=2$ and $D=3$ the mean-field curves are shifted the by the factors 9.56 and 2.12 to the left.

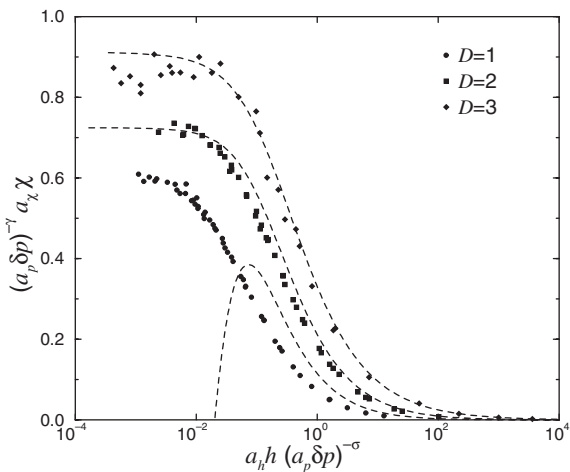


Fig. 8. The universal scaling function $\tilde{C}(1, x)$ for various dimensions. The dashed lines correspond to an ϵ -expansion of an RG approach.⁽²²⁾ The universal amplitude R_χ is obtained from the extrapolation $a_h h (a_p \delta p)^{-\sigma} \rightarrow 0$.

Similar to the universal amplitude ratios of the susceptibility and the fluctuations other universal combinations can be defined. Well known from equilibrium phase transitions is the quantity (see, e.g., ref. 4)

$$R_\chi = \Gamma d_c B^{\delta-1}, \quad (43)$$

which is also experimentally accessible, e.g., for magnetic systems. Here, Γ denotes the amplitude of the susceptibility χ in zero field ($\chi \sim \Gamma \delta T^{-\gamma}$) and B is the corresponding amplitude of the order parameter M ($M \sim B \delta T^\beta$). The factor d_c describes how the order parameter M depends on the conjugated field H at $\delta T = 0$ ($H \sim d_c M^\delta$).

In case of directed percolation these amplitudes correspond to the values $B = a_p^\beta \tilde{R}(1, 0)$, $\Gamma = a_p^\gamma a_h \tilde{C}(1, 0)$ as well as $d_c = a_h^{-1} \tilde{R}(0, 1)^{-\delta}$ where $\delta = \sigma/\beta$. The normalizations $\tilde{R}(1, 0) = \tilde{R}(0, 1) = 1$ yield for the amplitude combination

$$R_\chi = \tilde{C}(1, 0) \quad (44)$$

which is obviously a universal quantity. In Fig. 8 the scaling function $\tilde{C}(1, x)$ is plotted as a function of $x = a_h h (a_p \delta p)^{-\sigma}$ for $D = 1, 2, 3$. The corresponding data saturates for $x \rightarrow 0$. Our estimates are $R_\chi = 0.60 \pm 0.04$ for $D = 1$, $R_\chi = 0.72 \pm 0.04$ for $D = 2$, and $R_\chi = 0.86 \pm 0.08$ for $D = 3$. Note that the error-bars reflect only the data scattering in Fig. 8. In contrast to the amplitude $\tilde{C}(1, 0)/\tilde{C}(-1, 0)$ the data of R_χ are affected by the uncertainties of the exponent γ and the uncertainties of the metric factors a_p, a_h . These uncertainties increase the error-bars significantly. The two- and three-dimensional data agree quite well with the RG-results $R_\chi = 0.7244\dots$ for $D = 2$ and $R_\chi = 0.9112\dots$ for $D = 3$.⁽²²⁾ In the one-dimensional model the ϵ^2 -expansion yields again an unphysical result ($R_\chi = -3.927\dots$).

6. CONCLUSIONS

We considered the universal steady state scaling behavior of directed percolation with an external field in $D \geq 2$ dimensions. Our data for $D = 5$ coincide with the mean field solution, confirming that $D_c = 4$ is the upper critical dimension. At D_c we presented for the first time a numerical scaling analysis of DP including logarithmic corrections. Our results agree well with those of a recently performed RG approach.⁽²³⁾ Apart from the scaling functions we also considered amplitude ratios and combinations for the order parameter fluctuations and the susceptibility. A comparison with RG⁽²²⁾ results reveals that higher orders than $\mathcal{O}(\epsilon^2)$ are necessary to describe the scaling behavior in low dimensions.

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REFERENCES

1. H. E. Stanley, *Rev. Modern Phys.* **71**:S358 (1999).
2. E. A. Guggenheim, *J. Chem. Phys.* **13**:253 (1945).
3. S. Milošević and H. E. Stanley, *Phys. Rev. B* **6**:1002 (1972).
4. V. Privman, P. C. Hohenberg, and A. Aharony, in *Universal Critical-Point Amplitude Relations in Phase Transitions and Critical Phenomena, Vol. 14*, C. Domb and J. L. Lebowitz, eds. (Academic Press, London, 1991).
5. K. G. Wilson, *Phys. Rev. B* **4**:3174 (1971).
6. K. G. Wilson, *Phys. Rev. B* **4**:3184 (1971).
7. F. J. Wegner and E. K. Riedel, *Phys. Rev. B* **7**:248 (1973).
8. K. G. Wilson and J. Kogut, *Phys. Rep. C* **12**:75 (1974).
9. H. Hinrichsen, *Adv. Phys.* **49**:815 (2000).
10. H. K. Janssen, *Z. Phys. B* **42**:151 (1981).
11. P. Grassberger, *Z. Phys. B* **47**:365 (1982).
12. M. Rossi, R. Pastor-Satorras, and A. Vespignani, *Phys. Rev. Lett.* **85**:1803 (2000).
13. J. W. Essam, *J. Phys. A* **22**:4927 (1989).
14. J. L. Cardy and U. C. Täuber, *Phys. Rev. Lett.* **13**:4780 (1996).
15. I. Jensen, *Phys. Rev. Lett.* **70**:1465 (1993).
16. I. Jensen and R. Dickman, *Phys. Rev. E* **48**:1710 (1993).
17. S. Lübeck and R. D. Willmann, *J. Phys. A* **35**:10205 (2002).
18. M. A. Muñoz, G. Grinstein, R. Dickman, and R. Livi, *Phys. Rev. Lett.* **76**:451 (1996).
19. F. van Wijland, *Phys. Rev. Lett.* **89**:190602 (2002).
20. P. Grassberger and A. de la Torre, *Ann. Phys. (N.Y.)* **122**:373 (1979).
21. P. Pfeuty and G. Toulouse, *Introduction to the Renormalization Group and Critical Phenomena* (Wiley, Chichester, 1994).
22. H. K. Janssen, Ü. Kutbay, and K. Oerding, *J. Phys. A* **32**:1809 (1999).
23. H. K. Janssen and O. Stenull, *Phys. Rev. E* **69**:016125 (2004).
24. P. Grassberger, *Phys. Rev. E* **56**:3682 (1997).
25. P. Grassberger, R. Hegger, and L. Schäfer, *J. Phys. A* **27**:7265 (1994).
26. S. Lübeck, *Phys. Rev. E* **58**:2957 (1998).
27. D. V. Kytarev, S. Lübeck, P. Grassberger, and V. B. Priezzhev, *Phys. Rev. E* **61**:81 (2000).
28. L. Roters, S. Lübeck, and K. D. Usadel, *Phys. Rev. E* **66**:069901 (2002).
29. P. Grassberger, *Phys. Rev. E* **67**:036101 (2003).
30. S. Lübeck and P. C. Heger, *Phys. Rev. Lett.* **90**:230601 (2003).
31. A. A. Fedorenko and S. Stepanow, *Phys. Rev. E* **67**:057104 (2003).
32. H. K. Janssen and O. Stenull, *Phys. Rev. E* **68**:036131 (2003).

33. O. Stenull and H. K. Janssen, *Phys. Rev. E* **68**:036129 (2003).
34. E. Domany and W. Kinzel, *Phys. Rev. Lett.* **53**:311 (1984).
35. A. Y. Tretyakov and N. Inui, *J. Phys. A* **28**:3985 (1995).
36. P. Grassberger and Y.-C. Zhang, *Physica A* **224**:169 (1996).
37. V. Privman and M. E. Fisher, *Phys. Rev. B* **30**:322 (1984).
38. H. Mori and K. J. Mc Neil, *Prog. Theor. Phys.* **57**:770 (1977).
39. P. Schofield, *Phys. Rev. Lett.* **22**:606 (1969).
40. B. D. Josephson, *J. Phys. C* **2**:1113 (1969).
41. S. P. Obukhov, *Physica A* **101**:145 (1980).
42. J. L. Cardy and R. L. Sugar, *J. Phys. A* **13**:L423 (1980).
43. A. I. Larkin and D. E. Khmel'nitskiĭ, *JETP* **29**, 1123 (1969).
44. A. Aharony, *Phys. Rev. B* **8**:3363 (1973).
45. J. A. Griffiths, J. D. Litster, and A. Linz, *Phys. Rev. Lett.* **38**:251 (1977).
46. J. Bringmann, R. Courths, and H. J. Guggenheim, *Phys. Rev. Lett.* **40**:1286 (1978).
47. N. Aktekin, *J. Stat. Phys.* **104**:1397 (2001).
48. K. Binder and D. W. Heermann, *Monte Carlo Simulation in Statistical Physics* (Springer, Berlin, 1997).
49. I. Jensen, *J. Phys. A* **32**:5233 (1999).
50. C. A. Voigt and R. M. Ziff, *Phys. Rev. E* **56**:R6241 (1997).
51. I. Jensen, *Phys. Rev. A* **45**:R563 (1992).